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EFFECT OF PERIODIC OSCILLATIONS OF VELOCITY AND DENSITY OF A MEDIUM ON DISINTEGRATION OF LIQUID JETS*

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In a number of problems connected with the disintegration of liquid jets into drops (the atomization of liquids, the obtaining of emulsions, the transition of laminar flow of a liquid to turbulent, and so forth), considerable interest is attached to the study of the effect of the fluctuations of the flow velocity, that arise as a consequence of various causes, on the stability of the flows under consideration. It is of interest also to estimate theoretically the effect of density fluctuations of the gaseous medium surrounding the liquid jet on the disintegration of the jet. These fluctuations occur, for example, in the combustion chambers of liquid fuel jet engines and may change the conditions of the work process.

A theoretical investigation is presented of the effect of velocity and density fluctuations of the medium surrounding a cylindrical liquid jet on the disintegration of the jet. For the solution the method of small disturbances is applied.

It is shown that: (1) with fluctuations of the flow velocity of the liquid and of the density of the medium there is a change in the character and in the wave length of the unstable perturbations and instead of a single region (characteristic in the absence of fluctuations), an infinite number of separate unstable disturbances arise, (2) the optimum wave length is less than that for the case of absence of fluctuations, that is, the fluctuations of the flow velocity and of the density of the medium lead to a decrease in the droplet dimensions obtained in the jet disintegration, and (3) the obtained results of the theoretical analysis are qualitatively confirmed by the available experimental data.

We shall consider the stability of a circular cylindrical liquid jet. We choose a system of coordinates in which the jet is stationary while the surrounding medium moves with velocity U. The jet density

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and the density of the surrounding medium are denoted by ρ_1 and ρ_2, respectively. It is assumed that the velocity U and the density ρ_2 are periodic functions of the time. The fluids are assumed to be ideal and to possess potential flows.

1. The equation of the velocity potential in a cylindrical system of coordinates (r,φ,z) is

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1.1}$$

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As is known (refs. 1 and 2) the following boundary conditions are satisfied on the surface of the liquid jet:

(a) The normal velocities are equal (for $r = a$):

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial z}U = \frac{\partial\Phi_2}{\partial r}, \qquad \frac{\partial\zeta}{\partial t} = \frac{\partial\Phi_1}{\partial r} \tag{1.2}$$

where ζ is the radial perturbation of the jet surface and the subscripts 1 and 2 refer to the jet and the medium, respectively.

(b) The difference of the pressures in the jet and medium is equal to the pressure of the surface tension

$$p_1 - p_2 = -T\left(\frac{\zeta}{a^2} + \frac{1}{a^2}\frac{\partial^2\zeta}{\partial\varphi^2} + \frac{\partial^2\zeta}{\partial z^2}\right) \tag{1.3}$$

where T is the surface tension coefficient, and p_1 and p_2 the pressures for the perturbed motion, determined by the Lagrange-Cauchy integral

$$p_1 = -\rho_1\frac{\partial\Phi_1}{\partial t}, \qquad p_2 = -\rho_2\left(\frac{\partial\Phi_2}{\partial t} + \frac{\partial\Phi_2}{\partial z}U\right) \tag{1.4}$$

Solving equation (1.1) for the jet and the medium by the method of separation of variables we obtain

$$\Phi_1 = A(t)I_m(kr)e^{im\varphi+ikz}, \qquad \Phi_2 = B(t)K_m(kr)e^{im\varphi+ikz} \tag{1.5}$$

where $A(t)$ and $B(t)$ are functions of time, $k = 2\pi/\lambda$ the wave number, λ the length of the perturbation wave, propagated along the z-axis, m the number of waves over the circumference of the jet cross-section, and I_m and K_m Bessel functions of imaginary argument of order m.

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The radial perturbation of the surface from the initial position of the undisturbed jet may be presented in the form

$$\zeta = \overline{\zeta} e^{im\varphi + ikz} \tag{1.6}$$

Substituting expressions (1.4), (1.5) and (1.6) in the boundary conditions (1.2) and (1.3) we obtain

$$\overline{\zeta}' = A(t) I_m'(ka), \qquad \overline{\zeta}' + ikU\overline{\zeta} = B(t) K_m'(ka)$$

$$- \rho_1 A'(t) I_m(ka) + \rho_2 B'(t) K_m(ka) + \rho_2 ikUB(t) K_m(ka)$$

$$= \frac{\overline{\zeta}}{a^2} T(k^2a^2 + m^2 - 1) \tag{1.7}$$

where

$$I_m'(ka) = \frac{d}{dr} I_m(kr), \qquad K_m'(ka) = \frac{d}{dr} K_m(kr) \text{ for } r = a$$

Eliminating from this system the functions A and B and introducing the nondimensional magnitudes

$$M = \frac{\rho_2}{\rho_1}, \qquad \alpha = ka, \qquad L = U\sqrt{\frac{\rho_1 a}{T}}, \qquad \tau = t\sqrt{\frac{T}{\rho_1 a^3}} \tag{1.8}$$

we obtain the equation for $\overline{\zeta}(\tau)$

$$\overline{\zeta}''(MK - I) + 2i\alpha MLK\overline{\zeta}' - [\alpha^2 ML^2 K + \alpha(\alpha^2 + m^2 - 1) - i\alpha ML'K]\overline{\zeta} = 0 \tag{1.9}$$

where

$$I = \frac{I_m(\alpha)}{I_m'(\alpha)}, \qquad K = \frac{K_m(\alpha)}{K_m'(\alpha)} \tag{1.10}$$

and I_m' and K_m' are derivatives of the Bessel functions with respect to α.

We reduce equation (1.9) to the normal form

$$u'' + Ju = 0$$

where

$$u(\tau) = \bar{\zeta}(\tau) \exp\left\{\frac{1}{2}\int \frac{g}{f}\, d\tau\right\} \qquad \left(J = \frac{h}{f} - \frac{1}{4}\left(\frac{g}{f}\right)^2 - \frac{1}{2}\left(\frac{g}{f}\right)'\right) \tag{1.11}$$

J is the invariant of the differential equation, while

$$\begin{gathered} f = MK - I, \qquad g = 2i\alpha MLK \\ h = -[\alpha^2 ML^2 K + \alpha(\alpha^2 + m^2 - 1)] + i\alpha ML'K \end{gathered} \tag{1.12}$$

Thus

$$u(\tau) = \bar{\zeta}(\tau) \exp\left\{i\alpha K \int \frac{ML}{MK - I}\, d\tau\right\}$$

$$J = \frac{1}{(MK - I)^2}[\alpha^2 ML^2 KI - \alpha(\alpha^2 + m^2 - 1)(MK - I) + i\alpha KLM'I] \tag{1.13}$$

We shall consider separately the effect of the fluctuations of the velocity and density of the medium on the stability of the liquid jet with respect to small perturbations.

2. Let us consider the effect of the oscillations of the flow velocity (i.e., of the parameter L), on the disintegration of the jet, assuming the density of the medium to be constant (M = constant).

We present the nondimensional velocity L in the form

$$L = L_0[1 + \epsilon\psi(\nu\tau)] \qquad \left(\nu = \omega\sqrt{\frac{\rho_1 a^3}{T}}\right) \tag{2.1}$$

where ψ is a periodic function of period 2π, ϵ the relative amplitude of the velocity oscillations of the medium, which we assume small in comparison with unity, ν the nondimensional frequency of the superposed oscillations, and ω the circular frequency. We then obtain from equation (1.13)

$$u(\tau) = \bar{\zeta}(\tau) \exp\left\{\frac{i\alpha MK}{MK - I}\int L\, d\tau\right\}$$

$$J = \frac{1}{(MK - I)}[\alpha^2 MKIL_0^2 - \alpha(\alpha^2 + m^2 - 1)(MK - I) + 2\alpha^2 MKIL_0^2 \epsilon\psi(\nu\tau)] \tag{2.2}$$

Putting $\nu\tau = x$ we reduce equation (1.9) to the form

$$u'' + [\lambda + \gamma\psi(x)]u = 0 \tag{2.3}$$

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$$\lambda = \frac{1}{\nu^2(MK - I)^2}\left[\alpha^2 MKIL_o^2 - \alpha(\alpha^2 + m^2 - 1)(MK - I)\right], \qquad \gamma = \frac{2\alpha^2 MKIL_o^2\epsilon}{\nu^2(MK - I)^2} \tag{2.4}$$

Equation (2.3) is a Hill equation. The general solution for this equation (ref. 3) has the form

$$u = C_1 e^{\mu x}\varphi(x) + C_2 e^{-\mu x}\varphi(-x) \tag{2.5}$$

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where $\varphi(x)$ is a periodic function, μ a characteristic exponent depending on the parameters λ and γ and determining the character of the solution. The solutions of interest to us (increasing with time) correspond to $\mathrm{Re}[\mu] > 0$.

Let us first consider the very simple impulsive oscillations of the form

$$\psi(x) = \begin{cases} 1 & (-\pi \leq x < 0) \\ -1 & (0 \leq x < \pi) \end{cases} \tag{2.6}$$

In this case, the solution of equation (2.3) is expressed in terms of trigonometric functions, while the exponent μ is computed by the formula (ref. 4)

$$\mathrm{ch}\ 2\pi\mu = \cos x_1 \cos x_2 - \frac{1}{2}\left(\frac{x_1}{x_2} + \frac{x_2}{x_1}\right)\sin x_1 \sin x_2 \tag{2.7}$$

where

$$x_1 = \pi\sqrt{\lambda + \gamma}$$

$$x_2 = \pi\sqrt{\lambda - \gamma}$$

With the aid of this equation it is possible to construct the boundary curves between the regions of stable and unstable solutions corresponding to $\mathrm{ch}\ 2\pi\mu = \pm 1$. In figure 1 these curves have been constructed in the coordinates λ and γ and the regions are shown corresponding to the stable solutions (hatched areas).

The parameters λ and γ of the Hill equation depend, in this case, on both the magnitudes M, L_o, m, ν, ϵ and on the wave number α. Eliminating the magnitude α from equations (2.4) we construct the curve $\lambda(\gamma)$ for the chosen values of the preceding parameters (fig. 1).

The curve $\lambda(\gamma)$ starts from the origin and successively intersects all regions of instability. With increasing distance from the origin all increasing values of the wave numbers α correspond to points of the curve.

In order to investigate the effect of the velocity oscillations of the medium (or of the outflow velocity of the liquid) it is necessary to construct the curves of dependence of the increment of the oscillations of the perturbed motions on the wave number of the regions of unstable (increasing) oscillations.

Since with an accuracy up to a periodic factor

$$\overline{\zeta} \sim u \sim e^{\mu x} = e^{\mu \nu x} \tag{2.8}$$

the increment of the oscillations will be equal to

$$Z = \mu\nu \tag{2.9}$$

In figure 2 the curves $Z(\alpha)$ are constructed for two oscillation frequencies ν (0.6 and 2) for values of the parameters $M = 10^{-3}$, $m = 0$, $L_0 = 40$, and $\epsilon = 0.4$. The curve $Z(\alpha)$ is also constructed for the case of absence of velocity oscillations ($\epsilon = 0$).

As we see, this curve for $\epsilon = 0$ possesses a maximum at $\alpha \approx 1.3$. In accordance with the Raleigh hypothesis (refs. 1 and 2) the wave length of the most unstable disturbance is determined by the drop diameter.

In superimposing velocity oscillations the character of the disintegration of the jet changes as follows: a large (in principle infinitely large) number of regions of instability in the coordinates λ and γ appears. Each of these regions has a definite range of wave numbers (wave lengths) and a value of the maximum increment of oscillations Z.

If we assume that the dimension of the drops (formed as a result of the disintegration of the jet) is determined by the wave length corresponding to the maximum value of the increment over all regions of instability, then the dimensions of the drops decrease in the case of oscillations of the flow velocity (fig. 2).

Experimental data are available on the effect of the flow velocity oscillations on the liquid jet (ref. 6). It was found that these oscillations lead to a decrease in the drop dimensions; the mean diameter of the drops under the conditions of the experiment decreased approximately by one-half.

These experiments therefore qualitatively confirm the preceding theoretically derived conclusion.

Within the limits of the chosen magnitudes of the parameters the change of frequency of the oscillations ν causes a redistribution of the regions of instability with respect to the wave numbers (wave lengths); at the same time the magnitude of the maximum increment of the oscillations practically does not change.

3. We now proceed to investigate the effect of the density fluctuations of the medium (the magnitude M) on the disintegration of the jet, assuming the flow velocity to be constant (L = constant).

We shall here assume that the magnitude M is a harmonic function of the time

$$M = M_0(1 + \delta \cos \nu\tau) \tag{3.1}$$

where δ is the amplitude.

From equations (1.12) and (1.13) we obtain expressions for the invariant of the differential equation ($\nu\tau = 2x$)

$$J(x) = \theta_0 - 2\theta_1 \cos 2x - 2i\theta_2 \sin 2x \tag{3.2}$$

where

$$\theta_0 = \frac{4}{\nu^2(M_0K - I)^2}[\alpha^2 M_0 L^2 KI - \alpha(\alpha^2 + m^2 - 1)(M_0K - I)]$$

$$\theta_1 = -\frac{2M_0K\alpha\delta}{\nu^2(M_0K - I)^3}\left[\alpha L^2 I(1 - 2KM_0) + (\alpha^2 + m^2 - 1)(KM_0 - I)\right] \tag{3.3}$$

$$\theta_2 = -\frac{2\alpha M_0 KIL\delta}{\nu(M_0K - I)^2}$$

Equation (1.9) then assumes the form of a Hill equation

$$u'' + (\theta_0 - 2\theta_1 \cos 2x - 2i\theta_2 \sin 2x)u = 0 \tag{3.4}$$

This solution can likewise be presented in the form of equation (2.3).

Comparing these curves we see that for the same values of the parameters M and L, frequencies and amplitudes, the curves characterizing the density oscillations of the medium is more steep. As is shown by computations, the curves μ = constant in each region of instability of the diagram (fig. 3) are such that with increasing distance from the origin on the a-axis, the magnitudes of the exponent μ increase. This means that for the same conditions the values of the increments Z in the case of density oscillations of the medium will in absolute value be less than the values for the case of the flow velocity oscillations.

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It follows that the density oscillations of the medium surrounding the liquid jet have less effect on the disintegration of the jet than the oscillations of the flow velocity. If the curve $Z(\alpha)$, similar to the curve in figure 2, is constructed for this case the same effect of the density oscillations can be qualitatively established as that which holds for the velocity oscillations.

The preceding conclusions derived on the effect of the density oscillations of the medium are confirmed by the experiments published in reference 7.

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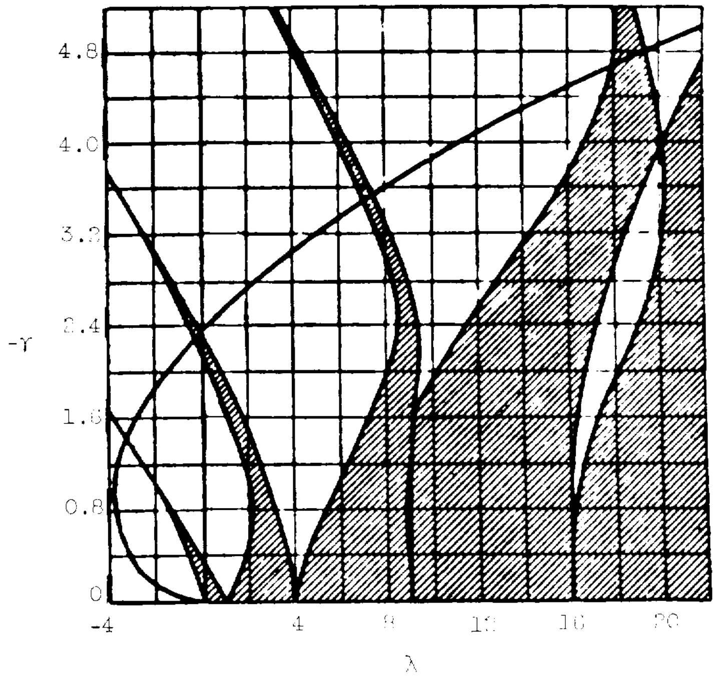

Figure 1.

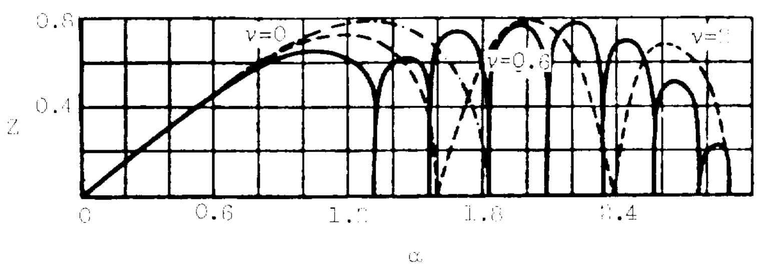

Figure 2.

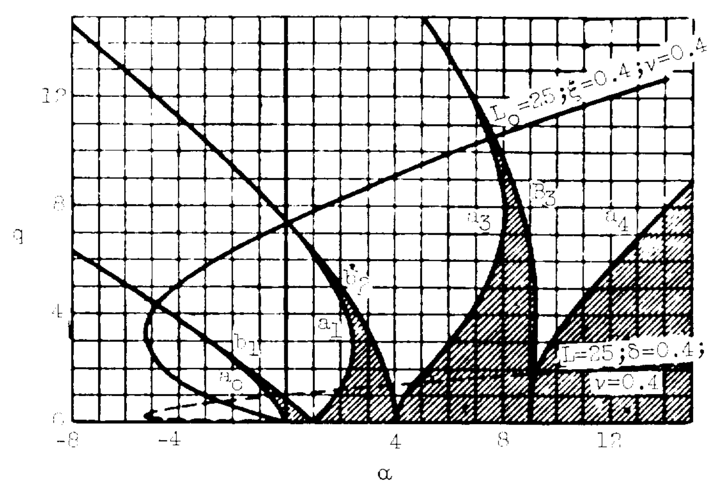

Figure 3.